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SECOND NOTE ON THE GROUPS GENERATED BY OPERATES TRANSFORMING EACH OTHER INTO THEIR INVERSES.

BY

G. A. MILLER.

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SECOND NOTE ON THE GROUPS GENERATED BY OPERATES TRANSFORMING EACH OTHER INTO THEIR INVERSES.

[*A Correction.*]

By G. A. Miller.

IN volume XL. (1909), p. 366, of this *Journal* I gave a brief note on the groups mentioned in the heading of the present paper. The main theorem of this note was stated inaccurately, as Schur has clearly pointed out in his recent review of it, *Jahrbuch ueber die Fortschritte der Mathematik*, volume XL. (1911), p. 189. The writer regrets especially the fact that he misunderstood a remark by Schur in an earlier review published in the *Fortschritte*,* and, as a consequence, attributed the incorrect theorem to Schur, who was nowise responsible for the error. There is some comfort, however, in the thought that the true theorem relating to the point in question seems even more interesting than the false one which it is to replace. The former of these two theorems, which will be proved presently, may be stated as follows:—*There is one and only one group of order* 2^n, $n>2$, *which can be generated by a set of* m *operators such that each of them transforms each of the* $m-1$ *remaining ones into its inverse and that no two of them are commutative. When* $n=3$ *this is the quaternion group and when* $n=4$ *it is the Hamiltonian group of order* 16, *but for larger values of* n *it cannot be Hamiltonian.*

Let $t_1, t_2, \ldots, t_m$ represent m operaters such that each of them transforms each of the $m-1$ remaining ones into its inverse and that no two of them are commutative. It is evident that the latter of these conditions may be replaced by one asserting that none of these m operators is of order two. Since any two non-commutative operators, which transform each other into their inverses, generate the quaternion group,† it results that each of the m operators $t_1, \ldots, t_m$ is of order 4 and that they all have the same square. It will be convenient to assume, in what follows, that t_α, $1<\alpha\leq m$, is not contained in the group generated by the $\alpha-1$ operators $t_1, \ldots, t_{\alpha-1}$. This assumption can clearly not affect the properties of the group generated by $t_1, \ldots, t_m$. On the other hand, it enables us to establish the relation $m=n-1$, which would evidently not exist necessarily without this assumption.

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* vol. 37 (1908), p. 171.

† *Quarterly Journal of Mathematics*, vol. XXXVII. (1906), p. 287.

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Since the group $\{t_1, t_2\}$ generated by the two operators t_1, t_2 is the quaternion, and $\{t_1, t_2, t_3\}$ is the Hamiltonian group of order 16,* we may assume, in what follows, that $m > 3$. From the facts that each of a set of generators of $\{t_1, \ldots, t_{\alpha-1}\}$ is transformed into its inverse by t_α and that t_α^2 is contained in $\{t_1, \ldots, t_{\alpha-1}\}$, it follows that the order of $\{t, \ldots, t_\alpha\}$ is twice the order of $\{t_1, \ldots, t_{\alpha-1}\}$. Hence $m = n - 1$, as was stated above.

Suppose that $\{t_1, \ldots, t_{\alpha-1}\}$ contains one and only one sub-group of index 2 which does not include any one of the operators $t_1, \ldots, t_{\alpha-1}$. This sub-group of order $2^{\alpha-1}$, together with the operator $t_\alpha t_\beta$, $1 \leqq \beta \leqq \alpha - 1$, which is of order 4, generate a group of order 2^α. This group does not contain any of the operators $t_1, \ldots, t_\alpha$. In fact, it would coincide with $\{t_1, \ldots, t_{\alpha-1}\}$ if it would contain any one of the operators $t_1, \ldots, t_{\alpha-1}$, and it would contain both t_α and t_β if it would contain t_α. Hence $\{t_1, \ldots, t_\alpha\}$ involves one sub-group of index 2, which includes none of the operaters $t_1, \ldots, t_\alpha$. This is clearly the only possible sub-group, with the given properties, contained in $\{t_1, \ldots, t_\alpha\}$, since such a sub-group cannot involve any one of the products obtained by multiplying t_α into an operator of the given sub-group of order $2^{\alpha-1}$. As the quaternion group $\{t_1, t_2\}$ contains one and only one sub-group of order 4 which does not include either of the operators t_1, t_2, we have proved, by complete induction, that *the group generated by* $t_1, \ldots, t_m$ *contains one and only one sub-group of index* 2 *which does not include one of these generators.*

It is now easy to prove that there is exactly one such group of order 2^n, where n is any natural number greater than 2. In fact, this follows almost directly from the theorem stated at the end of the preceding paragraph. If we assume that there is only one such group of order 2^α, which we may assume to be generated by $t_1, \ldots, t_{\alpha-1}$, then this group must contain one sub-group of order $2^{\alpha-1}$ composed of all its operators which are commutative with t_α. As this sub-group cannot involve any of the operators $t_1, \ldots, t_{\alpha-1}$ and is completely determined by this fact, it results that there is only one such group of order $2^{\alpha+1}$. Since there is only one such group of order 8, it follows therefore, by complete induction, that there is one and only one such group of order 2^n, $n > 2$.

G. A. Miller

Since the group H, generated by $t_1, \ldots, t_{\alpha-1}$, contains one and only one sub-group K of index 2, which does not involve any of the operators $t_1, \ldots, t_{\alpha-1}$, and since K involves all the

* *Loc. cit.*

operators of H which are commutative with each of the operators $t_a, \ldots, t_m$, it is very easy to find the orders of the operators of the groups under consideration. If we assume that all the operators of H are either of order 2 or of order 4 and that all these operators of order 4 have a common square, it results directly that the operators of the group of order 2^{a+1}, generated by t_a and H, have the same properties. Since the conditions imposed on H are satisfied when H is the quaternion group, it is proved, by complete induction, that *the group generated by* $t_1, \ldots, t_m$ *involves only operators of orders* 2 *and* 4, *and all of these operators of order* 4 *have a common square. This common square generates the commutator subgroup of the entire group.*

By means of these theorems it is very easy to see that the products obtained by multiplying the operators of order 4 in K by t_a are all of order 2, while the products obtained by multiplying t_a into any one of the remaining operators of K is of order 4. On the other hand, the order of the product obtained by multiplying t_a and any operator s of H, which is not also in K, is the same as the order s. These facts may be employed to determine the number of the operators of order 2, and hence also the number of those of order 4, in any one of the groups under consideration. In fact, starting with the case when H is the quaternion group it is clear that K is the cyclic group of order 4 and that the remaining operators of H are of order 4. Hence the group G_{16} obtained by extending H by means of $t_a = t_3$ contains three operators of order 2 and twelve of order 4.

If we let H represent this G_{16} it results that K is the quaternion group and that G_{32}, which is obtained by extending G_{16} by means of t_4, involves eleven operators of order 2 and twenty operators of order 4. If we let H represent this G_{32}, K will be the Hamiltonian group of order 16. As this involves operators of order 2, which are not commutative with t_4, and as every larger K involves all the preceding K's, it follows that every H whose order exceeds 16 involves non-invariant operators of order 2. In particular, *the group generated by* $t_1, \ldots, t_m$ *is not Hamiltonian whenever* $m > 3$.

To obtain a general formula for the number of the operators of order 2 in the group G generated by $t_1, \ldots, t_m$, we may observe that if we include, for brevity of statement, the identity among the operators of order 2, the K of G_{16} involves just half of the operators of order 2 which are contained in G_{16}. We may therefore start with an H of order 2^a, in which K includes m_1 operators of order 2, including the identity, while

H involves $2m_1$ such operators. The nine groups obtained by extending this H successively by means of $t_\alpha, \ldots, t_{\alpha+8}$ involve operators of order 2 in accord with the following table, the identity being included among these operators. These formulæ can easily be verified by means of the theorems established above.

Order of H.	Number of operators of order 2 in K.	Remaining operators of order 2 in H.
$2^{\alpha+1}$,	$2m_1$,	$2^{\alpha-1}$,
$2^{\alpha+2}$,	$2^{\alpha-1}+2m_1$,	$2^\alpha+2^{\alpha-1}-2m_1$,
$2^{\alpha+3}$,	$2^{\alpha+1}$,	$2^{\alpha+1}+2^\alpha-4m_1$,
$2^{\alpha+4}$,	$2^{\alpha+2}+2^\alpha-4m_1$,	$2^{\alpha+2}+2^\alpha-4m_1$,
$2^{\alpha+5}$,	$2^{\alpha+3}+2^{\alpha+1}-8m_1$,	$2^{\alpha+3}$,
$2^{\alpha+6}$,	$2^{\alpha+4}+2^{\alpha+1}-8m_1$,	$2^{\alpha+4}-2^{\alpha+1}+8m_1$,
$2^{\alpha+7}$,	$2^{\alpha+5}$,	$2^{\alpha+5}-2^{\alpha+2}+16m_1$,
$2^{\alpha+8}$,	$2^{\alpha+6}-2^{\alpha+2}+16m_1$,	$2^{\alpha+6}-2^{\alpha+2}+16m_1$,
$2^{\alpha+9}$,	$2^{\alpha+7}-2^{\alpha+3}+32m_1$,	$2^{\alpha+7}$.

From this table it is clear that if we start with any groud following one in which K involves just half the operatosr of order 2, including the identity, every fourth one of the successive groups will have this property. By means of this periodicity and the fact that $m_1=2$ when $\alpha=4$, it is not difficult to deduce the following general formulæ for the numbers of the operators of order 2 in the system of groups under consideration. If 2^n represents the order of the group G generated by $t_1, \ldots, t_m$, there will be exactly $2^{n-1}-1$ operators of order 2 in this group whenever n is of the form $4k+2$, $k>0$. Hence exactly half of the operators of such a group are of order 4 for these values of n. For the other values of n the number of operators of order 2 in G are as follows:

Form of n.	Number of operators of order 2.
$8k$, $k>0$,	$2^{8k-1}+2^{4k}-1$,
$8k+1$, $k>0$,	$2^{8k}+2^{4k}-1$,
$8k+3$,	$2^{8k+2}-2^{4k+1}-1$,
$8k+4$,	$2^{8k+3}-2^{4k+2}-1$,
$8k+5$,	$2^{8k+4}-2^{4k+2}-1$,
$8k+7$,	$2^{8k+6}+2^{4k+3}-1$.

Since the commutator sub-group of G is contained in each of its cyclic sub-groups of order 4, each of these cyclic sub-

groups is invariant under G. We proceed to prove that G has at most four invariant operators. This is included in the theorem that the K of any H has exactly two invariant operators. In fact, if we assume that the K of the H, generated by $t_1, \ldots, t_{a-1}$, involves only two invariant operators of H, it results at once that the K of the H, generated by $t_1, \ldots, t_a$, has the same property, since this K is obtained by adding to the preceding K the products of t_a into operators of $\{t_1, \ldots, t_{a-1}\}$, which are not in the K of this group. As none of these products is invariant under this group, we have proved that *the sub-group of index* 2 *under* G, *which does not involve any of the operators* $t_1, \ldots, t_m$, *contains two and only two invariant operators under* G.

Since G_{16} is the Hamiltonian group of order 16 it contains exactly four invariant operators, and from the theorem which has just been proved it follows that no G can contain more than four such operators. Suppose that the H of order 2^a involves four invariant operators. As only two of these are in the K of this H, it results that the H of order 2^{a+1} contains only two invariant operators, while the H of order 2^{a+2} contains four such operators. Hence we have the following theorem, by complete induction: *If the order of* G *is* 2^n *then* G *involves two or four invariant operators as* n *is odd or even. These invariant operators constitute a cyclic group when* n *is of the form* $4k+2$ *and only then.* This theorem furnishes proofs of the facts that G cannot be Hamiltonian whenever $n > 4$ and that exactly half the operators of G are of order 4 when n is of the form $4k+2$.

If a series of operators $s_1, s_2, \ldots, s_\lambda$ be given, and if it is known that each of these operators transforms each of the remaining $\lambda - 1$ into their inverses, then these operators clearly generate the Abelian group of order 2^m and of type $(1, 1, \ldots)$ in the case when each of them is of order 2 and only then. If the order of at least one of these operators exceeds 2, the orders of all the others must exceed 2, and hence all of them must be of order 4. That is, if a set of operators is such that each of them transforms all of the others into their inverses then either all of these operators are of order 2 or all of them are of order 4. Hence what precedes furnishes a complete solution of the possible groups which may be generated by operators having the given property.*

* Cf. Schur, *Crelle*, vol. 139 (1911), p. 198.

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